PI Lectures on General Relativity:

Problem Set #1:

1. (a) The metric of the Euclidean 3-space $\mathbb{R}^3$, written in Cartesian coordinates, is expressed by the line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

Use the standard coordinate transform to spherical coordinates $\{r, \theta, \varphi\}$ to write the Euclidean 3-space (flat) metric in spherical coordinates.

(b) Write the 2-sphere (curved) metric as an induced metric on the unit sphere

$$x^2 + y^2 + z^2 = 1$$

(We can think of this as an isometric embedding of S^2 into $\mathbb{R}^3$.)

(c) Extrapolate this procedure to write the metric for the 3-sphere.

(d) Can you write the S^3 metric in a more symmetric form?

2. Consider a 'half-equatorial' curve $\mathcal{C}$ in $\mathbb{R}^3$ parameterized by $\lambda \in [0, \pi]$ described by $x(\lambda) = \cos\lambda$, $y(\lambda) = \sin\lambda$, and $z(\lambda) = 0$.

(a) Find the length of $\mathcal{C}$ by direct evaluation in $\{x, y, z\}$ coordinates.

(b) Find the length of $\mathcal{C}$ by working in spherical $\{r, \theta, \varphi\}$ coordinates.

3. Geodesics on S^2:

(a) Write the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu{}_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta = 0$$

explicitly in the 2-sphere $\{\theta, \varphi\}$ coordinates. (Recall that Christoffel symbols $\Gamma^\mu{}_{\alpha\beta} \equiv \frac{1}{2}\, g^{\mu\gamma}\, (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta})$ where the greek indices run over θ and φ, we sum over γ, and $g^{\mu\gamma}$ is the inverse metric defined by $g_{\alpha\beta}\, g^{\beta\gamma} = \delta^\gamma_\alpha$.)
Are curves of constant θ geodesics? Are curves of constant φ geodesics?

(b) Use the ∂_φ symmetry to express the geodesics as an integral, characterized by a conserved quantity ('angular momentum') ℓ. From this, you can obtain a closed-form expression for $\varphi(\theta)$.

(c) If you have ready access to e.g. Mathematica, confirm numerically that all geodesics obtained above are great circles on the sphere.

4. Curvature of the S^2

(a) Using the 2-sphere metric from (1b) and the corresponding Christoffel symbols from (3a), evaluate the Riemann tensor

$$R_{\alpha\beta\gamma}{}^{\delta} = -\partial_\alpha \Gamma^\delta{}_{\beta\gamma} + \partial_\beta \Gamma^\delta{}_{\alpha\gamma} + \Gamma^\mu{}_{\alpha\gamma}\,\Gamma^\delta{}_{\beta\mu} - \Gamma^\mu{}_{\beta\gamma}\,\Gamma^\delta{}_{\alpha\mu}$$

Note that due to the symmetries of Riemann (in particular $R_{abc}{}^{d} = -R_{bac}{}^{d}$ and $R_{abcd} \equiv R_{abc}{}^{e}\, g_{ed} = -R_{abcd}$), you only need to evaluate $R_{\theta\varphi\theta}{}^{\varphi}$, and then generate the remaining non-zero terms by simple manipulations).

(b) Evaluate the Ricci tensor

$$R_{\alpha\gamma} = R_{\alpha\beta\gamma}{}^{\beta}$$

where β is summed over.

(c) Evaluate the Ricci scalar (a.k.a. the scalar curvature)

$$R = R_\alpha{}^\alpha = R_{\alpha\beta}\, g^{\alpha\beta}$$

(in the last expression, both α and β are summed over).