PI Lectures on General Relativity:

Problem Set #2:

1. The Schwarzschild metric has the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

We can learn a lot about the geometry by considering geodesics in the spacetime.

(a) Using the constants of motion E and L associated with ∂_t and ∂_φ, and the norm of the tangent vector to affinely-parameterized null, spacelike, or timelike geodesics denoted by $\kappa = 0, 1, -1$ respectively, show that the radial component of the geodesic equation can be recast in terms of a 1-d motion in effective potential defined by $\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = 0$, where:

$$V_{\text{eff}}(r) = -\frac{E^2 + \kappa}{2} + \kappa \, \frac{M}{r} + \frac{L^2}{2\,r^2} - \frac{L^2\,M}{r^3}$$

(b) Show that there is a *null* circular orbit (defined by r for which both $V'_{\text{eff}}(r) = 0$ and $V_{\text{eff}}(r) = 0$). Find the value of r at which such an orbit exists, and determine whether it is stable or unstable.

(c) Now consider *radial* null geodesics in the spacetime (i.e. those with $L = 0$). Sketch the light cones (delimited by ingoing and "outgoing" null geodesics) in the (t, r) plane.

(d) Show that $r = 2M$ (corresponding to the event horizon) is reached by ingoing light rays in finite affine parameter.

2. Anti de Sitter (AdS) spacetime is a maximally symmetric space of constant negative curvature. A 4-dimensional AdS can be realized as the surface

$$-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_3^2 = -\ell^2$$

inside a flat 5-dimensional 'spacetime' (with 2 times),

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \cdots + dX_3^2 \ .$$

(The scale ℓ is called the AdS radius.)

(a) By choosing the embedding $X_i(t, r, \theta, \varphi)$ appropriately, show that the metric of (global) AdS_4 can be written in a static, spherically symmetric form as

$$ds^2 = -\left(\frac{r^2}{\ell^2} + 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{\ell^2} + 1} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2\right)$$

(b) Show that radial timelike geodesics all 'oscillate' with the same period, passing through $r = 0$ infinitely many times. (In particular the effective potential is that of a simple harmonic oscillator.)

(c) Determine which class of geodesics (timelike/null/spacelike) can have circular orbits, and whether such orbits are stable or unstable.

(d) The boundary of AdS lies at infinite r. Find the metric on a constant-r surface, rescale this metric by $1/r^2$, and take the $r \to \infty$ limit to find the boundary metric. From this argue that the boundary is timelike. (This is crucially different from spacetimes which are asymptotically flat, where the boundary is null.)