The AdS/CFT Correspondence
PI *It from Qubit* Summer School: Mukund Rangamani

1 Lecture 2

Q1. The simplest black hole spacetime we can describe in an asymptotically AdS geometry is the famous BTZ black hole in three spacetime dimensions. Let us consider the global geometry, whose metric is parameterized as

$$ds^2 = -\frac{r^2 - r_+^2}{\ell_{\text{AdS}}^2}\, dt^2 + \frac{\ell_{\text{AdS}}^2\, dr^2}{r^2 - r_+^2} + r^2\, d\varphi^2\,. \tag{1.1}$$

There are various things we can do analytically in this geometry, which make it a useful intuition building example.

(i) Compute the temperature of this solution analyzing the Euclidean geometry obtained by analytically continuing the timelike Killing field $t \to i\, t_{\text{E}}$. Obtain the entropy of the solution using the Bekenstein-Hawking formula and thence the specific heat.

(ii) Having obtained the Euclidean geometry, argue that it is topologically is a filled two-torus $\mathbf{T}^2$ (i.e., a handlebody), with one non-contractible one cycle. Further, argue that there are two distinct solutions obtained upon analytic continuation.

(iii) The solution (1.1) satisfies Einstein's equations obtained from

$$\mathcal{S}_{grav} = \frac{1}{16\pi\, G_N^{(3)}} \int d^3x \sqrt{-g}\left(R + \frac{2}{\ell_{\text{AdS}}^2}\right) + \frac{1}{8\pi\, G_N^{(3)}} \int d^2x\, \sqrt{-\gamma}\, \left(K + \Lambda_\partial\right) \tag{1.2}$$

where we have included a boundary term involving the extrinsic curvature, the Gibbons-Hawking term, to ensure that Einstein's equations derive from a variational principle. The second contribution involving Λ_∂ is a finite counter-term which we fix to ensure that the action is finite. Compute the on-shell action for the BTZ spacetime and fix Λ_∂ to obtain a finite on-shell action.

(iv) Compare the action computed in the previous part with that for the thermal AdS geometry, viz., the spacetime obtained by setting $r_+ = i\,\ell_{\text{AdS}}$, $t \to i\, t_{\text{E}}$, and requiring $t_{\text{E}} = t_{\text{E}} + \beta$ in the metric

$$ds^2 = -\left(\frac{r^2}{\ell_{\text{AdS}}^2} + 1\right) dt^2 + \frac{dr^2}{\frac{r^2}{\ell_{\text{AdS}}^2} + 1} + r^2\, d\varphi^2\,. \tag{1.3}$$

Requiring that the temperature or Euclidean periodicity is the same, and interpreting the on-shell value of the action as the saddle point value of the thermal free energy, decide which of the two geometries dominates the thermal physics.

(v) How do spacelike geodesics behave in the BTZ spacetime. For simplicity you can first consider geodesics on the $t = 0$ slice. You should however be able to get a closed form expression for the curve in the spacetime.

(vi) Describe the geometry when $t_+ = i\,\ell_{\rm AdS}(1-\mu)$ for $\mu \in (0,1)$. Can you infer what the geodesics in the spatial sections look like?

Q2. Let us now compute how the BTZ black hole reacts to being perturbed. Take a free scalar field with some mass m whose action is given to be

$$\mathcal{S}_{matter} = \frac{1}{16\,\pi G_N^{(3)}} \int d^3x\,\sqrt{-g}\,\left(\frac{1}{2}\,\nabla_A\phi\,\nabla^A\phi - \frac{1}{2}\,m^2\,\phi^2\right) \tag{1.4}$$

Write down the wave equation for the scalar field in the BTZ geometry (1.1) and mode decompose it using the symmetries. We wish to treat the black hole as an open system, so we will solve the Schrödinger equation resulting from the above wave equation, with the following boundary conditions.

- The modes are ingoing at the horizon.
- The modes die off fast enough to be normalizable at infinity.

To get an operational sense of the ingoing boundary conditions, view the horizon as a one-way membrane and come up with a definition of right/left moving waves across this membrane.

Use these boundary conditions to determine the eigenvalues of the frequency; these are the quasinormal modes.

Q3. We can repeat much of problem 1 for the Schwarzschild-AdS$_{d+1}$ black hole both in planar and global geometries. The metrics in the two cases are:

$$\begin{aligned}
\text{Global}: \qquad & ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2\,d\Omega_{d-1}^2\,, \qquad f(r) = \frac{r^2}{\ell_{\rm AdS}^2} + 1 - \frac{r_+^{d-2}}{r^{d-2}}\left(1 + \frac{r_+^2}{\ell_{\rm AdS}^2}\right) \\
\text{Planar}: \qquad & ds^2 = \frac{\ell_{\rm AdS}^2}{z^2}\left(-f(z)\,dt^2 + \frac{dz^2}{f(z)} + d\mathbf{x}_{d-1}^2\right)\,, \qquad f(z) = 1 - \frac{z^d}{z_+^d}
\end{aligned} \tag{1.5}$$

Try to in particular,

(i) Work out the temperature and specific heat for the two solutions.

(ii) Understand the topology of the solution.