The AdS/CFT Correspondence
PI *It from Qubit* Summer School: Mukund Rangamani

1 Lecture 3

Q1. Consider the 'AdS brachistochrone curve' in planar Schwarzschild-AdS_5, which you can think of as a string dangling between two fixed boundary points, say at $t = 0$. What is the profile of the string? By comparing the length of this curve in Schwarzschild-AdS_5 relative to its value in AdS_5 obtain the expectation value of a boundary Wilson loop operator.

Soln # 1.

Let us work with the metric

$$ds^2 = \frac{\ell_{\text{AdS}}^2}{z^2}\left(-f(z)\,dt^2 + \frac{dz^2}{f(z)} + d\mathbf{x}_{d-1}^2\right) , \qquad f(z) = 1 - \frac{z^d}{z_+^d} \tag{1.1}$$

The string is at $t = 0$ and we can fix its end points to be at $x = \pm a$, taking $\mathbf{x}_{d-1} = (x, \mathbf{y}_{d-2})$. Then we simply need to find the profile of the string $x(z)$ or equivalently $z(x)$. The Nambu-Goto action for a string is so as to give a area minimization problem:

$$\mathcal{S}_{string} = \frac{1}{2\pi\,\alpha'}\int d^2\xi\sqrt{-\gamma_{AB}\partial_a X^A\,\partial_b X^B} \tag{1.2}$$

where ξ^a are intrinsic coordinates on the string. Choosing $\xi^0 = t$ and $\xi^1 = x$, so that we end up with the undetermined classical variable $z(t,x) = z(x)$. Evaluate the induced metric:

$$ds^2_{string} = \frac{\ell_{\text{AdS}}^2}{z^2}\left(-f(z)\,dt^2 + \left(\frac{z'(x)^2}{f(z)} + 1\right)dx^2\right) \tag{1.3}$$

which leads to the action:

$$\begin{aligned}\mathcal{S}_{string} &= \frac{\ell_{\text{AdS}}^2}{2\pi\,\alpha'}\int dt\,dx\,\frac{1}{z(x)^2}\,\sqrt{z'(x)^2 + f(z)}\\ &= T\sqrt{\lambda}\int dx\,\frac{1}{z(x)^2}\,\sqrt{z'(x)^2 + f(z)}\,, \qquad \sqrt{\lambda} \equiv \frac{\ell_{\text{AdS}}^2}{2\pi\,\alpha'}\end{aligned} \tag{1.4}$$

The action has a conserved Hamiltonian, which is obtained by a Legendre transform:

$$\frac{1}{z(x)^2}\frac{f(z)}{\sqrt{z'(x)^2 + f(z)}} = C \tag{1.5}$$

The rest of the analysis is straightforward.

Q2. Let us get some intuition for holographic entanglement entropy in a variety of situations. Most of these exercises are designed to be done analytically, though you can also attempt to do them numerically (and thence generalize). The task is to compute $S_{\mathcal{A}}(|\,\psi\rangle)$ where I will prescribe the global state $|\,\psi\rangle$ of the CFT, the boundary geometry and the region $\mathcal{A}$ below:

(i) $|\psi\rangle = |0\rangle$ on $\mathbb{R}^{d-1,1}$ and the region is a disc at $t=0$.

$$\mathcal{A}_\circ = \{(t,\xi,\Omega_{d-2})|t=0\,,\ \Omega_{d-2}:\text{arbitrary}\,,\ 0\leq \xi \leq R\}$$

(ii) $|\psi\rangle = |0\rangle$ on $\mathbb{R}^{d-1,1}$ and the region is a strip at $t=0$.

$$\mathcal{A}_\| = \{(t,x,\mathbf{y}_{d-2})|t=0\,,\ -L\leq y_i\leq L\,,\ -w\leq x\leq w\}$$

(iii) $|\psi\rangle = |0\rangle$ on $\mathbb{R}\times\mathbf{S}^{d-1}$ and the region is the polar-cap cutting the sphere $\mathbf{S}^{d-1}$ at a latitude around the north pole (which is set to be $\theta=0$)

$$\mathcal{A}_{polar} = \{(t,\theta,\Omega_{d-2})|-\theta_0<\theta<\theta_0\}\ \text{ where }\ d\Omega_{d-1}^2 = d\theta^2+\sin^2\theta\, d\Omega_{d-2}^2$$

(iv) Thermal state on $\mathbb{R}^{d-1,1}$ and the region is a disc at $t=0$, $\mathcal{A}_\circ$.

(v) Thermal state on $\mathbb{R}^{d-1,1}$ and the region is a strip at $t=0$, $\mathcal{A}_\|$.

(vi) Get explicit results for the answers in $d=2$ in various cases and compare with CFT_2 computations described in other lectures.

(vii) **Bonus 1:** As a more interesting situation, consider the thermofield double representation of the thermal state. The dual geometry is the eternal Schwarzschild-AdS_{d+1} black hole. Take the region $\mathcal{A}$ to be the union of half-spaces on both CFTs, i.e.,

$$\mathcal{A} = \{(t,x,\mathbf{y}_{d-2})_\text{L}\cup(t,x,\mathbf{y}_{d-2})_\text{R}\ |\ x_\text{R}\geq 0\,, x_\text{L}\geq 0\}$$

(viii) **Bonus 2:** Take a CFT_2 at finite temperature on a circle. Let us give a can consider a Gibbs state where in addition to finite T we also include a finite chemical potential for angular momentum. Can you work out the answer for the entanglement for $\mathcal{A}$ being an arc of the spatial circle.

Soln # 2.

(i) Take the metric on the plane to be in polar coordinates adapted to the symmetries. The action for the minimal surface is then:

$$\mathcal{S} = 4\pi\, c_\text{eff}\,\omega_{d-2}\int d\xi\,\frac{\xi^{d-2}}{z^{d-1}}\sqrt{1+z'(\xi)^2}\,. \tag{1.6}$$

Here $\omega_{d-2} = \frac{2\,\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$ is the area of a unit $\mathbf{S}^{d-2}$. Convince yourself that the equations of motion are solved by the hemisphere:

$$z^2+\xi^2 = R^2 \tag{1.7}$$

which leads to (use a trigonometric parameterization as $z(\theta)$ and $\xi(\theta)$ to derive the following)

$$S_{\mathcal{A}_\circ} == \frac{2}{\pi^{\frac{d}{2}-1}} \frac{\Gamma(\frac{d}{2})}{d-2} a_d \frac{\text{Area}(\partial\mathcal{A})}{\epsilon^{d-2}} + \cdots + \begin{cases} 4\,(-1)^{\frac{d}{2}-1}\, a_d \, \log \frac{2R}{\epsilon}\,, & d = 2m\,, \\ (-1)^{\frac{d-1}{2}}\, 2\pi\, a_d\,, & d = 2m+1\,. \end{cases} \tag{1.8}$$

(ii) The action we want is

$$\begin{aligned} \mathcal{S} &= 4\pi\, c_{\text{eff}} \int d^{d-2}x\, dx_1\, \frac{\sqrt{1+z'(x_1)^2}}{z^{d-1}} \\ \delta\mathcal{S} = 0 &\Longrightarrow z'(x_1) = \frac{\sqrt{z_*^{2(d-1)} - z^{2(d-1)}}}{z^{d-1}}\,, \qquad z_* = a\, \frac{\Gamma\left(\frac{1}{2(d-1)}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d}{2(d-1)}\right)} \end{aligned} \tag{1.9}$$

One can solve the for the surface explicitly in terms of hypergeometric functions; we give the expression for the two lobes of the surface $x_1 > 0$ and $x_1 < 0$ which smoothly meet at $x_1 = 0, z = z_*$:

$$\pm x_1(z) = \frac{z^d}{d\, z_*^{d-1}}\, {}_2F_1\left(\frac{1}{2}, \frac{d}{2(d-1)}, \frac{3d-2}{2d-2}, \left(\frac{z}{z_*}\right)^{2(d-1)}\right) - \frac{\sqrt{\pi}}{d} \frac{\Gamma\left(\frac{3d-2}{2d-2}\right)}{\Gamma\left(\frac{2d-1}{2(d-1)}\right)} \tag{1.10}$$

which leads to

$$S_{\mathcal{A}_\parallel} = \frac{4\pi\, c_{\text{eff}}}{d-2}\, L^{d-2} \left[\frac{2}{\epsilon^{d-2}} - \left(\frac{2}{z_*}\right)^{d-1} \frac{1}{a^{d-2}}\right] \tag{1.11}$$

(iii) For the polar cap region we have the action for the minimal surface:

$$\mathcal{S} = 4\pi\, c_{\text{eff}}\, \omega_{d-2} \int d\xi\ \, (r \sin\theta)^{d-2} \sqrt{\frac{1}{f(r)} \left(\frac{dr}{d\xi}\right)^2 + r^2 \left(\frac{d\theta}{d\xi}\right)^2} \tag{1.12}$$

with an appropriate choice of $f(r) = 1 + \frac{r^2}{\ell_{\text{AdS}}^2}$.

(iv) For the thermal state on the disc

$$\mathcal{S}_\circ = 4\pi\, c_{\text{eff}}\, \omega_{d-2} \int d\xi\, \frac{\xi^{d-2}}{z^{d-1}} \sqrt{1 + \frac{z'(\xi)^2}{f(z)}}\,, \qquad f(z) = 1 - \frac{z^d}{z_+^d} \tag{1.13}$$

(v) For the thermal state on the strip:

$$\mathcal{S}_\parallel = 4\pi\, c_{\text{eff}} L^{d-2} \int dx_1\, \frac{1}{z^{d-1}} \sqrt{1 + \frac{z'(x_1)^2}{f(z)}}\,, \qquad f(z) = 1 - \frac{z^d}{z_+^d} \tag{1.14}$$

(vi) For the CFT_2 on the plane in the vacuum state one gets explicit expression:

$$S_{\mathcal{A}} = 4\pi\, c_{\text{eff}}\, 2 \int_{\frac{\epsilon}{a}}^{\frac{\pi}{2}} \frac{d\xi}{\sin\xi} = 8\pi\, c_{\text{eff}} \log\frac{2a}{\epsilon} = \frac{c}{3}\,\log\frac{2a}{\epsilon} \tag{1.15}$$

where we used the Brown-Henneaux result again.

The corresponding result for a region on a finite spatial circle can be also directly computed (from the polar-cap expression for instance) to be

$$S_{\mathcal{A}} = \frac{c}{3}\,\log\left(\frac{\ell_{\mathbf{S}^1}}{\pi\,\epsilon}\sin\left(\frac{2a}{\ell_{\mathbf{S}^1}}\right)\right) \tag{1.16}$$

We translated the answer in terms of the arc-length a of the region ($\ell_{\mathbf{S}^1}$ is the proper radius of the circle).

The thermal state of the CFT_2 on non-compact space $x \in \mathbb{R}$ is described by the planar BTZ geometry

$$ds^2 = -\frac{(r^2 - r_+^2)}{\ell_{\text{AdS}}^2}\,dt^2 + \frac{dr^2}{r^2 - r_+^2} + \frac{r^2}{\ell_{\text{AdS}}^2}\,dx^2 \tag{1.17}$$

The extremal surface satisfies:

$$\frac{dr}{dx} = \frac{r}{\ell_{\text{AdS}}^2}\sqrt{(r^2 - r_+^2)\left(\frac{r^2}{r_*^2} - 1\right)}\,, \qquad r_* = r_+\,\coth(a\,r_+) \tag{1.18}$$

where r_* is determined by restricting the range of $x \in (-a, a)$. We can compute its length and obtain the answer for the entanglement entropy:

$$S_{\mathcal{A}} = \frac{c}{3}\,\log\left(\frac{\beta}{\pi\,\epsilon}\,\sinh\left(\frac{2a}{\beta}\right)\right) \tag{1.19}$$

To write the answer in this form, we used the fact that BTZ black hole of radius r_+ corresponds to a thermal state of the field theory at $T = \frac{r_+}{2\pi\,\ell_{\text{AdS}}^2}$.

The thermal state on $\mathbf{S}^1$ is trickier, since one has to be careful with the homology constraint (see supplementary reading). One finds:

$$S_{\mathcal{A}} = \begin{cases} \frac{c}{3}\,\log\left(\frac{\beta}{\pi\,\epsilon}\,\sinh\left(\frac{R}{\beta}\,\varphi_{\mathcal{A}}\right)\right), & \varphi_{\mathcal{A}} < \varphi_{\mathcal{A}}^\star \\ \frac{c}{3}\,\pi\,r_+ + \frac{c}{3}\,\log\left(\frac{\beta}{\pi\,\epsilon}\,\sinh\left(\frac{R}{\beta}\,(\pi - \varphi_{\mathcal{A}})\right)\right), & \varphi_{\mathcal{A}} \geq \varphi_{\mathcal{A}}^\star \end{cases} \tag{1.20}$$

where we wrote the answer for a spatial circle of size R. We also introduced the critical angular scale $\varphi_{\mathcal{A}}^\star$ where the two saddles of the area functional exchange dominance; explicitly

$$\varphi_{\mathcal{A}}^\star(r_+) = \frac{1}{r_+}\,\coth^{-1}\left(2\,\coth(\pi\,r_+) - 1\right)\,, \qquad \lim_{r_+\to\infty}\,\varphi_{\mathcal{A}}^\star(r_+) = \pi\,. \tag{1.21}$$