It from Qubit: Entanglement Theory Tuesday Problems

Measures of entanglement

(a) The *Schmidt number* of a bi-partite pure state is defined as the number of non-zero coefficients in the state's Schmidt decomposition. Prove that for transformations among pure states, the Schmidt number is an entanglement monotone, i.e., that it is non-increasing under local operations and classical communication (LOCC).

(b) The *global robustness of entanglement* of a multi-partite state ρ is defined as the smallest probabilistic weight s with which one can mix the state ρ with some state τ such that the mixture $\frac{1}{1+s}\rho + \frac{s}{1+s}\tau$ is a separable state. Formally, it is

$$R_G(\rho) = \arg\inf_s \{\frac{1}{1+s}\rho + \frac{s}{1+s}\tau \in \text{Separable}\}. \quad (1)$$

Show that R_G is an entanglement monotone, i.e., show that if $\rho \to \sigma$ under LOCC, then $R_G(\sigma) \leq R_G(\rho)$.

Hint: The proof makes use of the following two facts about an LOCC operation: (1) by virtue of being LOCC, when acting on a separable state, it yields another separable state, and (2) by virtue of being a linear map, it can be distributed over a mixture.

Bound entanglement

(a) Let the two pure states forming a product state on AB be called the A-marginal and B-marginal respectively. For a product state to be orthogonal to all five elements of the UPB, either its A-marginal must be orthogonal to at least three of the A-marginals of the UPB, or its B-marginal must be orthogonal to at least three of the B-marginals of the UPB. But any three A-marginals or B-marginals of the UPB spans the full three-dimensional space of that party, preventing any new vector from being orthogonal to all the existing ones.

(b) The mixed state in question is

$$\rho = \frac{1}{4}\left(I - \sum_k |\psi_k\rangle\langle\psi_k|\right). \quad (2)$$

This is clearly non-separable because there are no product states in its support. To show that it is not distillable, one makes use of the fact that having negative partial transpose is a necessary condition for being distilable (because LOCC operations preserve the property of being PPT and a Bell pair is NPT). The state in question has positive partial transpose (PPT) by virtue of the fact that, denoting partial tranpose on B by T_B, we have that $I^{T_B} = I$ and $|\psi_k\rangle\langle\psi_k|^{T_B}$ is a valid product state. Consequently,

$$\rho^{T_B} = \frac{1}{4}\left(I^{T_B} - \sum_k |\psi_k\rangle\langle\psi_k|^{T_B}\right) \geq 0. \quad (3)$$

Measures of entanglement

(a) This is easily proven using Nielsen's theorem together with the fact that one vector cannot majorize another vector if the latter has a greater number of nonzero components. A more explicit proof is as follows: Consider a given step in the LOCC protocol wherein one of the parties (say Alice) implements a local operation. Suppose the Kraus operators of her operation are $\{K_\mu\}$. If the bipartite state prior to the operation is $|\psi\rangle$ and after the operation is $|\phi\rangle$, then we must have $K_\mu \rho_\psi K_\mu^\dagger = \rho_\phi$ for all μ, where ρ_ψ is the reduced density operator for $|\psi\rangle$ on Alice's side. The rank of ρ_ϕ cannot be greater than the rank of ρ_ψ. Given that the Schmidt number is equal to the rank of the reduced density operator, we see that the Schmidt number cannot increase under LOCC.

(b) If $\rho \to \sigma$, then there is an LOCC operation, denoted $\mathcal{E}$, such that $\sigma = \mathcal{E}(\rho)$. Now note that for a given s and τ, if the state $\frac{1}{1+s}\rho + \frac{s}{1+s}\tau$ is separable, then because LOCC operations take separable states to separable states, the image of $\frac{1}{1+s}\rho + \frac{s}{1+s}\tau$ under $\mathcal{E}$ is also separable. But by linearity, this image is $\frac{1}{1+s}\mathcal{E}(\rho) + \frac{s}{1+s}\mathcal{E}(\tau)$. Recalling that $\sigma = \mathcal{E}(\rho)$, it follows that $R_G(\sigma) \leq s$. Applying this argument for the s and τ that achieve the minimum in the expression for $R_G(\rho)$, we obtain $R_G(\sigma) \leq R_G(\rho)$.