It from Qubit Summer School Solutions (Tensor Networks)

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1 Matrix Product State

1.b MPS with $\chi = 1$ for each of the product states

$$\begin{aligned}
|prod_1\rangle &= |00\cdots 0\rangle\,, && (1)\\
&\to \mathbf{v}_l = \mathbf{v}_r = (1),\quad A_0 = (1),\quad A_1 = (0), && (2)\\
|prod_2\rangle &= |11\cdots 1\rangle\,, && (3)\\
&\to \mathbf{v}_l = \mathbf{v}_r = (1),\quad A_0 = (0),\quad A_1 = (1), && (4)\\
|prod_3\rangle &= |\phi\phi\cdots\phi\rangle\,, \qquad |\phi\rangle \equiv a\,|0\rangle + b\,|1\rangle\,, && (5)\\
&\to \mathbf{v}_l = \mathbf{v}_r = (1),\quad A_0 = (a),\quad A_1 = (b), && (6)\\
|prod_4\rangle &= \left|\phi^\perp\phi^\perp\cdots\phi^\perp\right\rangle, \quad \left|\phi^\perp\right\rangle \equiv b^*\,|0\rangle - a^*\,|1\rangle\,, && (7)\\
&\to \mathbf{v}_l = \mathbf{v}_r = (1),\quad A_0 = (b^*),\quad A_1 = (-a^*). && (8)
\end{aligned}$$

1.c MPS with $\chi = 2$ for the generalized GHZ state

$$\begin{aligned}
\left|GHZ'\right\rangle &= \alpha\,|\phi\phi\cdots\phi\rangle + \beta\left|\phi^\perp\phi^\perp\cdots\phi^\perp\right\rangle && (9)\\
\to \mathbf{v}_l &= \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \qquad \mathbf{v}_r = \begin{pmatrix}1\\ 1\end{pmatrix}, && (10)\\
A_0 &= \begin{pmatrix}a & 0\\ 0 & b^*\end{pmatrix}, \qquad A_1 = \begin{pmatrix}b & 0\\ 0 & -a^*\end{pmatrix}. && (11)
\end{aligned}$$

1.d MPS with $\chi = 2$ for the W state,

$$\begin{aligned}
|W\rangle &\equiv \frac{1}{\sqrt{N}}\left(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots + |00\cdots 1\rangle\right) && (12)\\
\mathbf{v}_l &= \frac{1}{\sqrt{N}}\begin{pmatrix}0\\ 1\end{pmatrix}, \qquad \mathbf{v}_r = \begin{pmatrix}1\\ 0\end{pmatrix}, && (13)\\
A_0 &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad A_1 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, && (14)
\end{aligned}$$

and for the generalized W state

$$\begin{aligned} \left|W'\right\rangle &\equiv \alpha\left|00\cdots0\right\rangle + \beta\left|W\right\rangle \qquad (15) \\ \rightarrow \mathbf{v}_l &= \begin{pmatrix} \alpha \\ \frac{\beta}{\sqrt{N}} \end{pmatrix}, \qquad \mathbf{v}_r = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (16) \\ A_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad (17) \end{aligned}$$

2 Tree Tensor Network

2.b TTN for the product states $|prod_1\rangle, |prod_2\rangle, |prod_3\rangle, |prod_4\rangle$ of Eqs. 1-7.

$$\begin{aligned} |prod_1\rangle &= |00\cdots0\rangle, \qquad (18) \\ &\rightarrow v = |0\rangle, \quad A = |00\rangle\langle 0|, \qquad (19) \\ |prod_2\rangle &= |11\cdots1\rangle, \qquad (20) \\ &\rightarrow v = |0\rangle, \quad A = B = \cdots = |00\rangle\langle 0|, \quad C = |11\rangle\langle 0|, \qquad (21) \\ |prod_3\rangle &= |\phi\phi\cdots\phi\rangle, \qquad |\phi\rangle \equiv a|0\rangle + b|1\rangle, \qquad (22) \\ &\rightarrow v = |0\rangle, \quad A = B = \cdots = |00\rangle\langle 0|, \quad C = |\phi\phi\rangle\langle 0|, \qquad (23) \\ |prod_4\rangle &= \left|\phi^\perp\phi^\perp\cdots\phi^\perp\right\rangle, \qquad \left|\phi^\perp\right\rangle \equiv b^*|0\rangle - a^*|1\rangle, \qquad (24) \\ &\rightarrow v = |0\rangle, \quad A = B = \cdots = |00\rangle\langle 0|, \quad C = \left|\phi^\perp\phi^\perp\right\rangle\langle 0|, \qquad (25) \\ & \qquad (26) \end{aligned}$$

2.c TTN for the GHZ state $|GHZ\rangle$ and generalized GHZ state $|GHZ'\rangle$:

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}\left(|00\cdots0\rangle + |11\cdots1\rangle\right), \qquad (27) \\ &\rightarrow v = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \quad A = |00\rangle\langle 0| + |11\rangle\langle 1| \qquad (28) \end{aligned}$$

$$\begin{aligned} \left|GHZ'\right\rangle &= \alpha|\phi\phi\cdots\phi\rangle + \beta\left|\phi^\perp\phi^\perp\cdots\phi^\perp\right\rangle, \qquad (29) \\ &\rightarrow v = \alpha|0\rangle + \beta|1\rangle, \quad A = B = \cdots = |00\rangle\langle 0| + |11\rangle\langle 1|, \qquad (30) \\ & \qquad C = |\phi\phi\rangle\langle 0| + \left|\phi^\perp\phi^\perp\right\rangle\langle 1|, \qquad (31) \end{aligned}$$

2.d TTN for the product of entangled pairs

$$\begin{aligned} |pairs_1\rangle &\equiv |\psi_{12}\rangle \otimes |\psi_{34}\rangle \otimes |\psi_{56}\rangle \otimes |\psi_{78}\rangle, \qquad (32) \\ |\psi_{AB}\rangle &\equiv \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right). \qquad (33) \\ &\rightarrow v = |0\rangle, \quad A = B = |00\rangle\langle 0|, \quad C = |\psi_{i,i+1}\rangle\langle 0|. \qquad (34) \end{aligned}$$

2.e Trivial coarse-graining:

$$\rightarrow C = |00\rangle\langle 1| + |01\rangle\langle 2| + |10\rangle\langle 3| + |11\rangle\langle 4|. \tag{35}$$

3 Multi-scale entanglement renormalization ansatz

4 Scaling of entanglement in an MPS

4.b The tensor network for ρ_L can be split into two pieces by just breaking two bond indices. The rank of ρ_L is thus at most χ^2.

5 Scaling of correlations in an MPS

Notice that $c(x,y)$ is a sum of at most χ^2 exponentials

$$c(x,y) = \sum_{\alpha=1}^{\chi^2} c_\alpha (\lambda_\alpha)^{|x-y|-1} = \sum_{\alpha=1}^{\chi^2} c'_\alpha e^{-|x-y|/\xi_\alpha}, \qquad c'_\alpha \equiv c_\alpha/\lambda_\alpha,\ \xi_\alpha \equiv 1/\log(\lambda_\alpha). \tag{36}$$

At large distances, the contribution from the λ_α with largest real part and such that $c_\alpha \neq 0$ (denoted λ below) will dominate over the rest of terms in the sum, and we obtain

$$c(x,y) \approx e^{-\frac{|x-y|}{\xi}}, \qquad \xi \equiv \frac{-1}{\log(\lambda)}. \tag{37}$$