Entanglement entropy and c-function for a massive scalar in two dimensions

The aim of this problem is to compute numerically the entanglement entropy for a free massive scalar field in vacuum in two space-time dimensions. You will be able to extract the c-function from the values of the entanglement entropy.

The discretized Hamiltonian of a massive scalar in two spacetime dimensions reads

$$\mathcal{H} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(\pi_n^2 + (\phi_{n+1}-\phi_n)^2 + m^2\phi_n^2\right)\,. \tag{1}$$

We have set the lattice spacing ϵ to one.

You will need the field and momentum vacuum correlators to compute the entropy. These are given by the following expressions

$$X_{ij} = \langle \phi_i \phi_i \rangle = f(i-j) = \int_{-\pi}^{\pi} dx \frac{e^{ix(i-j)}}{4\pi\sqrt{m^2 + 2(1-\cos(x))}}\,, \tag{2}$$

$$P_{ij} = \langle \pi_i \pi_j \rangle = g(i-j) = \int_{-\pi}^{\pi} dx \frac{1}{4\pi} e^{ix(i-j)} \sqrt{m^2 + 2(1-\cos(x))}\,. \tag{3}$$

a) Choose a value of the mass $m \sim 1/100$ and compute a table with the correlation functions $f(i), g(i)$ for a few hundred sites. Note that we are aiming for the continuum limit of the theory. What does this requires for the mass values we can use?

b) Compute the entropy of an interval of size R. For this take the $R \times R$ correlator matrices X_R, P_R in R consecutive lattice points. Then compute the entropy with the formulas for Gaussian states in terms of correlation functions:

$$C_R = \sqrt{X_R P_R}\,, \tag{4}$$

$$S(R) = \mathrm{tr}\left((C_R + 1/2)\log(C_R + 1/2) - (C_R - 1/2)\log(C_R - 1/2)\right)\,. \tag{5}$$

(Warning: some eigenvalues may be very near 1/2, and numerical error can give complex values in (5). Notice you can simply choose to eliminate these eigenvalues rather than computing them with higher precision.)

c) Plot the entropies $S(R)$ as a function of the interval size. You should be able to see the following features

i) The entropy saturates to a constant value for $Rm \gg 1$. With taking $Rm \sim 3$ and $m \sim 1/10$ should be enough to see it. By changing the size of the mass you can check that the saturation constant is of the form

$$S \sim -\frac{1}{3}\log(m) + \mathrm{const}\,. \tag{6}$$

Considering that this formula reads in the continuum $S \sim -1/3\log(m\epsilon)$, and that the dependence on ϵ must be the same for small and large intervals, can you explain this formula?

ii) In the opposite limit $Rm \ll 1$ we expect to have the conformal result for a field with central charge $c = 1$

$$S(R) \sim \frac{1}{3}\log(R) + \mathrm{const}\,. \tag{7}$$

However, you should find some surprise trying to get this. Try with a very small (i.e. $m \sim 10^{-10}$, but non-zero!) mass and $R \sim (10-50)$. For this small mass you should need to increase the working precision of the integrals for the correlators (say to 20-30 digits).

iii) Evaluate the c-function $C(Rm) = \frac{RdS(R)}{dR}$ and check that it converges to a limit in the continuum limit and that it is always decreasing, interpolating between 1/3 for small Rm (as you have seen this limit is hard to get in detail) and zero for large Rm.