Quantum NP and the complexity of ground states

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Problem set for qubit Summer school, 2016

1. We defined in the lecture that a language L is in BQP if there is a family $\mathcal{F} = \{C_n\}$ of polynomial size quantum circuits (made of say two-qubit unitary gates) s.t.:

   - every circuit C_n has an input x of $|x| = n$ bits and $m = O(poly(n))$ additional ancilla qubits initialized to $|0\rangle$
   - the output of the computation is considered to be the outcome of the measurement on a designated ancila qubit.
   - the size of the circuits, $|C_n|$, grows polynomially in n.
   - there is a polynomial time Turing Machine that on input 1^n outputs C_n (this is the condition we glossed over, and you can essentially ignore it for this problem set)
   - and most importantly
     - if $x \in L$, and $|x| = n$ then $\Pr(C_n(x) = 1) \geq 2/3$
     - if $x \notin L$, and $|x| = n$ then $\Pr(C_n(x) = 0) \geq 2/3$

   Show that BQP with error probability 1/3 equals BQP with error probability 2^{-n} using the following:

   (a) The majority function of n bits can be computed classically efficiently.

   (b) Any classical computation can be done on a quantum computer with the same number of gates (as shown in Watrous's course).

   (c) The Chernoff bound: Let $b_1, ...b_n \in \{0, 1\}$ be i.d.d. random variables. Let $B = \sum_i b_i$ and $\mu = E[B]$ be the expected value of B. Then for any $0 < \delta < 1$,

   $$\Pr(B \geq (1 + \delta)\mu) \leq e^{-\frac{\delta^2 \mu}{3}} \tag{1}$$

2. We mentioned in the lecture two types of constraints on the locality of our physical systems: They can either involve only k-local interactions, but with no geometrical constraints (call it non-geometrical locality), or there could actually be geometrical constraints, say the qubits are set on a 1-Dim lattice and interactions are allowed only between nearest neighbors.

Show that the problem of simulating a quantum circuit with a non-geometrical nocality can be reduced (using a classical efficient computation) to the problem of simulating a quantum circuit which is geometrically constrainted to be in one dimensions. (hint: the 1D system is allowed to use any two qubit gates as long as they are applied on nearest neighbors. Use the gate which swaps two qubits.)

3. consider the very simple one local Hamiltonian acting on $n+1$ qubits, defined as $H = \Pi_{n+1}$ where $\Pi = |-\rangle\langle-|$ (and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$), and Π_{n+1} means that this projection is applied to the right most qubit. (The Hamiltonian does nothing to the other qubits). Consider the $n+1$ qubit state for arbitrary complex n qubit states $|\alpha\rangle$ and $|\beta\rangle$. What is the expectation value of the energy with respect to H of the state

$$\frac{1}{\sqrt{2}}(|\alpha\rangle \otimes |0\rangle + |\beta\rangle \otimes |1\rangle) \tag{2}$$

Express this energy as a function of the inner product between $|\alpha\rangle$ and $|\beta\rangle$.

4. Consider a very simple quantum circuit consiting of just one two qubit gate, which acts on two qubits initialized to the state $|0\rangle|\psi\rangle$, where ψ is unknown. The gate is defined as the unitary $|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|01\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ (we actually need not to specify what the gate does to the other basis states for this question).

   (a) What is the history state of the circuit (using one qubit to count the time passing from 0 to 1)?

   (b) Apply the circuit to Hamiltonian construction (using one qubit for the clock) to find a Hamiltonian whose ground state is the history state of the circuit.

   (c) What is the ground energy of this Hamilonian? What is its ground state?

   (d) Is the Hamiltonian frustrated?

5. Repeat question 3 except now the gate is changed: $|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and $|01\rangle \rightarrow \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

   (d) Calculate the probability of the right qubit to be 1 for any given input state $|\psi\rangle$.

(e) Argue why the Hamiltonian is frustrated. (as a bonus, write it as a Hermitian matrix, and find its ground state numerically.)

6. Recall the following two assumptions:

   (a) the quantum Church Turing thesis (which is strongly believed to hold), which states that "any physically realizable computational model can be simulted by a quantum computer efficiently (with polynomial overhead).

   (b) And another assumption which is strongly believed to hold, coming from intuition regarding hard problems, is that a quantum computer will need exponential time to solve a QMA hard problem. (this is a strengthening of the assumption that QMA is strictly larger than BQP).

   Recall also the result mentioned in class (the proof of which we did not see), which states that the local Hamiltonian problems for 2-local interactions between particles set on a line, where the interactions are only between nearest neighbors, is QMA-complete. (for this result we use 8 dimensional particles - which is higher than 2, but still a constant).

   Using the above, argue why there should exist one dimensional physical systems set on the line (with nearest neighbor interactions only) for which the relaxation time to the ground state will take time which is exponential in the size of the system.